B.Sc I.

S.S. College J. Bad
Deptt - Maths
Period - 11 A.M to 12 P.M

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Topic - Symmetric Functions
(Theory of Equation)

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For the Cubic $x^3+px^2+qx+r=0$ find the value of the following Symmetric functions

(i) $\Sigma(\alpha+\beta-\gamma)^3$ (ii) $\Sigma \frac{\beta^2+\alpha^2}{\beta+\alpha}$

(iii) $\Sigma \frac{\alpha^2+\beta^2}{\alpha\beta}$

Solution - Here given Cubic Equation is

$$x^3+px^2+qx+r=0$$

$\Rightarrow \Sigma\alpha=-p \qquad \Sigma\alpha\beta=q \qquad \alpha\beta\gamma=-r$

(1) $\Sigma(\alpha+\beta-\gamma)^3$

$$=(\alpha+\beta-\gamma)^3+(\beta+\gamma-\alpha)^3+(\gamma+\alpha-\beta)^3$$

$$=(\alpha+\beta+\gamma-2\gamma)^3+(\alpha+\beta+\gamma-2\alpha)^3+(\alpha+\beta+\gamma-2\beta)^3$$

$$=(-p-2\gamma)^3+(-p-2\alpha)^3+(-p-2\beta)^3$$

$$=-\left[(p^3+6p^2\gamma+12p\gamma^2+8\gamma^3)+(p^3+8\alpha^3+6p^2\alpha+12p\alpha^2)+(p^3+8\beta^3+6p^2\beta+12p\beta^2)\right]$$

$$=-\left[3p^3+8(\alpha^3+\beta^3+\gamma^3)+6p^2(\alpha+\beta+\gamma)+12p(\alpha^2+\beta^2+\gamma^2)\right]$$

$$=-\left[3p^3+8\Sigma\alpha^3+6p^2\Sigma\alpha+12p\Sigma\alpha^2\right]$$

$$=-\left[3p^3+8(3pq-p^3-3r)+6p^2(-p)+12p(p^2-2q)\right]$$

$$=-\left[3p^3+\cancel{24pq}-8p^3-24r-6p^3+12p^3-\cancel{24pq}\right]$$

$$=-\left[p^3-24r\right]=24r-p^3$$

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(11) $\sum \frac{\beta^2+\alpha^2}{\beta+\alpha} = \frac{\alpha^2+\beta^2}{\alpha+\beta} + \frac{\beta^2+\gamma^2}{\beta+\gamma} + \frac{\gamma^2+\alpha^2}{\gamma+\alpha}$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha+\beta} + \frac{(\beta+\gamma)^2 - 2\beta\gamma}{\beta+\gamma} + \frac{(\gamma+\alpha)^2 - 2\gamma\alpha}{\gamma+\alpha}$$

$$= \left[\alpha+\beta - \frac{2\alpha\beta}{\alpha+\beta}\right] + \left[\beta+\gamma - \frac{2\beta\gamma}{\beta+\gamma}\right] + \left[\gamma+\alpha - \frac{2\gamma\alpha}{\gamma+\alpha}\right]$$

$$= 2(\alpha+\beta+\gamma) - 2\left[\frac{\alpha\beta}{\alpha+\beta} + \frac{\beta\gamma}{\beta+\gamma} + \frac{\gamma\alpha}{\gamma+\alpha}\right]$$

$$= -2p - 2\left[\frac{\alpha\beta(\beta+\gamma)(\gamma+\alpha) + \beta\gamma(\alpha+\beta)(\gamma+\alpha) + \gamma\alpha(\alpha+\beta)(\beta+\gamma)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}\right]$$

$$= -2p - 2\left[\frac{\alpha\beta(-p-\alpha)(-p-\beta) + \beta\gamma(-p-\gamma)(-p-\beta) + \gamma\alpha(-p-\gamma)(-p-\alpha)}{(-p-\alpha)(-p-\beta)(-p-\gamma)}\right]$$

$$= -2p - 2\left[\frac{\alpha\beta\{p^2 + p(\alpha+\beta) + \alpha\beta\} + \beta\gamma\{p^2 + p(\beta+\gamma) + \beta\gamma\} + \alpha\gamma\{p^2 + p(\alpha+\gamma) + \alpha\gamma}{-\{p^3 + p^2(\alpha+\beta+\gamma) + p(\alpha\beta+\beta\gamma+\gamma\alpha) + \alpha\beta\gamma\}}\right]$$

$$= -2p - 2\left[\frac{p^2(\alpha\beta+\beta\gamma+\alpha\gamma) + p(\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \alpha^2\gamma + \alpha\gamma^2) + (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)}{-[p^3 - p^3 + pq - r]}\right]$$

$$= -2p - 2\frac{\left[p^2q + p\sum\alpha^2\beta + \sum\alpha^2\beta^2\right]}{-(pq-r)}$$

$$= -2p - 2\frac{\left[p^2q + p(3r - pq) + q^2 - 2pr\right]}{r - pq}$$

$$= \frac{2p^2q - 2pr - 2p^2q - 2p(3r - pq) - 2q^2 + 4pr}{r - pq}$$

$$= \frac{2p^2q - 4pr - 2q^2}{r - pq}$$

(iii) $$\sum\frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha}$$

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$$

$$= \frac{\alpha+\beta}{\gamma} + \frac{\alpha+\gamma}{\beta} + \frac{\beta+\gamma}{\alpha}$$

$$= \frac{-\gamma-p}{\gamma} + \frac{-\beta-p}{\beta} + \frac{-\alpha-p}{\alpha}$$

$$= -1 - \frac{p}{\gamma} - 1 - \frac{p}{\beta} - 1 - \frac{p}{\alpha}$$

$$= -3 - p\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) = -3 - p\frac{\sum\alpha\beta}{\alpha\beta\gamma}$$

$$= -3 - p\left(-\frac{q}{r}\right) = -3 + \frac{pq}{r}$$

$$= \frac{pq}{r} - 3.$$

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If $\alpha, \beta, \gamma, \delta$ be the roots of the Equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the values of the following symmetric Functions

(i) $\Sigma\alpha^2$

Here given equation is

$$x^4 + px^3 + qx^2 + rx + s = 0$$

$\Rightarrow \Sigma\alpha = -p$

$\Sigma\alpha\beta = q$

$\Sigma\alpha\beta\gamma = -r$

$\alpha\beta\gamma\delta = s$

(i) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= (-p)^2 - 2q$

$= p^2 - 2q$

(ii) $\Sigma\alpha^2\beta = \alpha^2(\beta+\gamma+\delta) + \beta^2(\alpha+\gamma+\delta) + \gamma^2(\alpha+\beta+\delta)$

$+ \delta^2(\alpha+\beta+\gamma)$

clearly the No of Terms will be ${}^4P_2 = \frac{\lfloor 4}{\lfloor 2} = 12$

Now $\Sigma\alpha^2\beta$ can be obtained by Multiplying $\Sigma\alpha$ and $\Sigma\alpha\beta$. the product will have $4 \times 6 = 24$ Terms out of which 12 are $\Sigma\alpha^2\beta$ and rest are of the Type $\alpha\beta\gamma$.

Hence $\Sigma\alpha \; \Sigma\alpha\beta = (\alpha+\beta+\gamma+\delta)(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$$= \underset{12}{\Sigma\alpha^2\beta} + 3(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$$

$$= \underset{12}{\Sigma\alpha^2\beta} + \underset{3\times4}{3\,\Sigma\alpha\beta\gamma}$$ (digit below Σ denote the No of Terms)

$$\therefore \Sigma\alpha^2\beta = \Sigma\alpha \; \Sigma\alpha\beta - 3\Sigma\alpha\beta\gamma$$

$$= (-p)(q) - 3(-r)$$

$$= 3r - pq$$

(iii) $\underset{4\times3}{\Sigma\alpha^2\beta\gamma} = \alpha^2(\beta\gamma+\beta\delta+\gamma\delta)+\beta^2(\alpha\gamma+\alpha\delta+\gamma\delta)$

$$+\gamma^2(\alpha\beta+\alpha\delta+\beta\delta)+\delta^2(\alpha\beta+\alpha\gamma+\beta\gamma)$$

Since $\Sigma\alpha^2\beta\gamma$ will be obtained by Multiplying $\Sigma\alpha$ and $\Sigma\alpha\beta\gamma$. the No of Terms in the product should be $4\times4=16$ the No of Terms in $\Sigma\alpha^2\beta\gamma$ be 12 and rest ar $\Sigma\alpha\beta\gamma\delta$ i.e 4

$$\therefore \Sigma\alpha \; \Sigma\alpha\beta\gamma = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$$

$$= \Sigma\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta$$

$$\Rightarrow \Sigma\alpha^2\beta\gamma = \Sigma\alpha \; \Sigma\alpha\beta\gamma - 4\alpha\beta\gamma\delta$$

$$= (-p)(-r) - 4\cdot s$$

$$= pr - 4s$$

And $\sum \alpha^2\beta^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2$

Hence $(\sum \alpha\beta)^2 = (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^2$

$$= \sum \alpha^2\beta^2 + 6\alpha\beta\gamma\delta + 2\sum \alpha^2\beta\gamma.$$

$$\therefore \sum \alpha^2\beta^2 = (\sum \alpha\beta)^2 - 2\sum \alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$$

$$= q^2 - 6s - 2(pr - 4s)$$

$$= q^2 - 2pr + 2s.$$

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