

B.Sc I.

S.S. college J. Bad
dept - Maths
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Topic - Symmetric Functions
(Theory of Equations)

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WhatsApp No -

For the Cubic $x^3 + px^2 + qx + r = 0$
Find the value of the following symmetric functions

(i) $\sum (\alpha + \beta - r)^3$

(ii) $\sum \frac{\beta^2 + \alpha^2}{\beta + \alpha}$

(iii) $\sum \frac{\alpha^2 + \beta^2}{\alpha\beta}$

Solution - Here given Cubic Equation is

$$x^3 + px^2 + qx + r = 0$$

$$\Rightarrow \sum \alpha = -p \quad \sum \alpha\beta = q \quad \alpha\beta r = -r$$

$$\begin{aligned} \textcircled{1} \quad \sum (\alpha + \beta - r)^3 &= (\alpha + \beta - r)^3 + (\beta + r - \alpha)^3 + (r + \alpha - \beta)^3 \\ &= (\alpha + \beta + r - 2r)^3 + (\alpha + \beta + r - 2\alpha)^3 + (\alpha + \beta + r - 2\beta)^3 \\ &= (-p - 2r)^3 + (-p - 2\alpha)^3 + (-p - 2\beta)^3 \\ &= - \left[(p^3 + 6p^2r + 12pr^2 + 8r^3) + (p^3 + 8\alpha^3 + 6p^2\alpha + 12p\alpha^2) \right. \\ &\quad \left. + (p^3 + 8\beta^3 + 6p^2\beta + 12p\beta^2) \right] \\ &= - \left[3p^3 + 8(\alpha^3 + \beta^3 + r^3) + 6p^2(\alpha + \beta + r) + 12p(\alpha^2 + \beta^2 + r^2) \right] \\ &= - \left[3p^3 + 8 \sum \alpha^3 + 6p^2 \sum \alpha + 12p \sum \alpha^2 \right] \\ &= - \left[3p^3 + 8(3pq - p^3 - 3r) + 6p^2(-p) + 12p(p^2 - 2q) \right] \\ &= - \left[3p^3 + 24pq - 8p^3 - 24r - 6p^3 + 12p^3 - 24pq \right] \\ &= - \left[p^3 - 24r \right] = 24r - p^3 \end{aligned}$$

$$(11) \sum \frac{p^2 + x^2}{p+x} = \frac{q^2 + p^2}{q+p} + \frac{p^2 + r^2}{p+r} + \frac{r^2 + x^2}{r+x}$$

$$= \frac{(q+p)^2 - 2qx}{q+p} + \frac{(p+r)^2 - 2pr}{p+r} + \frac{(r+x)^2 - 2rx}{r+x}$$

$$= \left[\frac{q+p - \frac{2qx}{q+p}}{q+p} \right] + \left[\frac{p+r - \frac{2pr}{p+r}}{p+r} \right] + \left[\frac{r+x - \frac{2rx}{r+x}}{r+x} \right]$$

$$= 2(q+p+r) - 2 \left[\frac{qx}{q+p} + \frac{pr}{p+r} + \frac{rx}{r+x} \right]$$

$$= -2p - 2 \left[\frac{qx(p+r)(r+x) + pr(q+p)(r+x) + rx(q+p)(p+r)}{(q+p)(p+r)(r+x)} \right]$$

$$= -2p - 2 \left[\frac{qx(-p-x)(-p-r) + pr(-p-r)(-p-q) + rx(-p-r)(-p-x)}{(-p-x)(-p-r)(-p-q)} \right]$$

$$= -2p - 2 \left[\frac{qx \{ p^2 + p(q+r) + xr \} + pr \{ p^2 + p(q+r) + rq \} + rx \{ p^2 + p(q+r) + qx \}}{(-p-x)(-p-r)(-p-q)} \right]$$

$$= -2p - 2 \left[\frac{-p^3 + p^2(q+r+x) + p(qx + pr + rx) + xpr}{(-p-x)(-p-r)(-p-q)} \right]$$

$$= -2p - 2 \left[\frac{p^2(xq + pr + rx) + p(x^2q + xq^2 + p^2r + pr^2 + x^2r + xq^2) + (x^2p^2 + pr^2 + rx^2)}{(-p-x)(-p-r)(-p-q)} \right]$$

$$= -2p - 2 \left[\frac{-p^3 - p^3 + p^2q - r}{(-p-x)(-p-r)(-p-q)} \right]$$

$$= \frac{-2p - 2 \left[p^2q + p \sum \alpha^2 \beta + \sum \alpha^2 \beta^2 \right]}{-(pq - r)}$$

$$= \frac{-2p - 2 \left[p^2q + p(3r - pq) + q^2 - 2pr \right]}{r - pq}$$

$$= \frac{2p^2q - 2pr - 2p^2q - 2p(3r - pq) - 2q^2 + 4pr}{r - pq}$$

$$= \frac{2p^2q - 4pr - 2q^2}{r - pq}$$

$$(11) \quad \sum \frac{\alpha^2 + \beta^2}{\alpha \beta} = \frac{\alpha^2 + \beta^2}{\alpha \beta} + \frac{\beta^2 + r^2}{\beta r} + \frac{r^2 + \alpha^2}{r \alpha}$$

$$= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{r} + \frac{r}{\beta} + \frac{r}{\alpha} + \frac{\alpha}{r}$$

$$= \frac{\alpha + \beta}{r} + \frac{\alpha + r}{\beta} + \frac{\beta + r}{\alpha}$$

$$= \frac{-r - p}{r} + \frac{-\beta - p}{\beta} + \frac{-\alpha - p}{\alpha}$$

$$= -1 - \frac{p}{r} - 1 - \frac{p}{\beta} - 1 - \frac{p}{\alpha}$$

$$= -3 - p \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{r} \right) = -3 - p \frac{\sum \alpha \beta}{\alpha \beta r}$$

$$= -3 - p \left(-\frac{q}{r} \right) = -3 + \frac{pq}{r}$$

$$= \frac{pq}{r} - 3$$

If $\alpha, \beta, \gamma, \delta$ be the roots of the Equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the values of the following Symmetric Functions

~~(1) $\sum \alpha^2$~~

Here given Equation is

$$x^4 + px^3 + qx^2 + rx + s = 0$$

$$\Rightarrow \sum \alpha = -p$$

$$\sum \alpha\beta = q$$

$$\sum \alpha\beta\gamma = -r$$

$$\alpha\beta\gamma\delta = s$$

$$\begin{aligned} \textcircled{1} \sum \alpha^2 &= (\sum \alpha)^2 - 2 \sum \alpha\beta \\ &= (-p)^2 - 2q \\ &= p^2 - 2q \end{aligned}$$

$$\begin{aligned} \textcircled{11} \sum \alpha^2\beta &= \alpha^2(\beta + \gamma + \delta) + \beta^2(\alpha + \gamma + \delta) + \gamma^2(\alpha + \beta + \delta) \\ &\quad + \delta^2(\alpha + \beta + \gamma) \end{aligned}$$

clearly the no of terms will be $4P_2 = \frac{4 \times 3}{1 \times 2} = 12$

Now $\sum \alpha^2\beta$ can be obtained by multiplying $\sum \alpha$ and $\sum \alpha\beta$. The product will have $4 \times 6 = 24$ terms out of which 12 are $\sum \alpha^2\beta$ and rest are of the type $\alpha\beta\gamma$.

Hence $\sum x \sum x y = (x+y+z+s)(x/y + x/r + x/s + y/r + y/s + z/s)$

$$= \sum x^2 y + 3(x/y r + x/y s + x/r s + y/r s)$$

$$\stackrel{12}{=} \sum x^2 y + 3 \sum x y r \quad \left(\begin{array}{l} \text{digit below } \sum \\ \text{denote the no of terms} \end{array} \right)$$

$$\begin{aligned} \sum x^2 y &= \sum x \sum x y - 3 \sum x y r \\ &= (-p)(9) - 3(-r) \\ &= 3r - p9 \end{aligned}$$

(iii) $\sum x^2 y r$ 4×3 $= x^2(yr + ys + rz) + y^2(xr + xs + rz) + r^2(xy + xs + yr)$

Since $\sum x^2 y r$ will be obtained by multiplying $\sum x$ and $\sum x y r$. the No of Terms in the product should be $4 \times 4 = 16$
the No of Terms in $\sum x^2 y r$ be 12 and rest are $\sum x y r s$ i.e 4

$$\begin{aligned} \therefore \sum x \sum x y r &= (x+y+z+s)(x/y r + x/s r + y/r s + y/s r + z/s r) \\ &= \sum x^2 y r + 4 x y r s \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum x^2 y r &= \sum x \sum x y r - 4 x y r s \\ &= (-p)(-r) - 4 \cdot 8 \\ &= pr - 48 \end{aligned}$$

Ans $\sum \alpha^2 \beta^2 = \alpha^2 \beta^2 + \alpha^2 \gamma^2 + \alpha^2 \delta^2 + \beta^2 \gamma^2 + \beta^2 \delta^2 + \gamma^2 \delta^2$

Hence $(\sum \alpha \beta)^2 = (\alpha \beta + \alpha \gamma + \alpha \delta + \beta \gamma + \beta \delta + \gamma \delta)^2$
 $= \sum \alpha^2 \beta^2 + 6\alpha \beta \gamma \delta + 2 \sum \alpha^2 \beta \gamma$

$\therefore \sum \alpha^2 \beta^2 = (\sum \alpha \beta)^2 - 2 \sum \alpha^2 \beta \gamma + 6\alpha \beta \gamma \delta$
 $= 9^2 - 68 - 2(p\gamma - 48)$
 $= 9^2 - 2p\gamma + 28$